1 Deterministic Asset Prices

[Final 2001] Consider the following deterministic Lucas fruit tree economy. There is a single representative agent who values consumption streams according to $\sum_{t=0}^{\infty} \beta^t u(c_t)$. There are (measure) $K_i > 0$ fruit trees of type $i \in \{1, 2\}$ which yield exogenous dividend streams of $d_{it} > 0$ units of the good. Trees are traded at (endogenous) prices $p_{it}$. The household also receives an endowment stream $e_t$.

(a) Solve the household problem using Dynamic Programming. Note that the household’s budget constraint is

$$e_t + \sum_{i=1}^{2} (p_{it} + d_{it}) k_{it} = c_t + \sum_{i=1}^{2} p_{it} k_{i,t+1}$$

where $k_{it}$ denotes the amount of type $i$ trees held by the household in period $t$.

(b) Define a competitive equilibrium.

(c) Solve in closed form for the prices of all trees in terms of (future) endowments and dividends. Hint: Note that the marginal rate of substitution terms $\alpha_{t+j} = u'(c_{t+j})/u'(c_t)$ are essentially exogenous because agents eat essentially exogenous amounts in each period in equilibrium.

(d) Suppose that endowments and dividends cycle. The endowments cycle between $e_t = e^{\text{odd}}$ in odd periods and $e_t = e^{\text{even}}$ in even periods. Assume $e^{\text{odd}} < e^{\text{even}}$. Tree 1 pays $d_{1t} = d$ in odd periods and nothing in even periods. Tree 2 pays $d_{2t} = d$ in even periods and nothing in odd periods. Calculate the asset price ratio, $p_{1t}/p_{2t}$, for odd and even $t$. Explain what you find. If you can’t solve this, explain what you would expect to find.

1.1 Answer: Deterministic Asset Prices

(a) The Bellman equation is

$$V(k_1, k_2) = \max v \left( e + \sum_{i=1}^{2} (p_{i} + d_{i}) k_{i} - \sum_{i=1}^{2} p_{i} k'_{i} \right) + \beta V(k'_1, k'_2)$$

The Euler equation is standard:

$$u'(c) = \beta R_i u'(c')$$

where $R_i = (p_{i} + d_{i})/p_{i}$ is the rate of return of a type $i$ tree. A solution consists of sequences $c_t$ and $a_t = \sum p_{it} k_{it}$ which satisfy the Euler equation and the flow budget constraint (and a TVC). Note that the portfolio composition is indeterminate.

(b) A competitive equilibrium is a set of sequences $(c_t, k_{it}, p_{it})$ that satisfy: 2 household conditions; $k_{it} = K_{it}$; $c_t = e_t + \sum d_{it} k_{it}$. Finally, for asset markets to clear, all trees must yield the same rate of return: $R_{it} = R_t$.

(c) This is standard iterating over a difference equation stuff:

$$p_{it} = \beta \alpha_{t+1} (d_{it+1} + p_{it+1})$$

$$= \sum_{j=1}^{\infty} \beta^j \alpha_{t+j} d_{it+j}$$

(1)
where $\alpha_{t+j} = u'(c_{t+j})/u'(c_t)$ is the marginal rate of substitution (which is essentially exogenous).

(d) Note that $\alpha_{t+j}$ now takes on only 2 values. If $t$ is even, then $c_t = e_{\text{even}} = e_{\text{even}} + dK_2$. If $t$ is odd, then $c_t = e_{\text{odd}} = e_{\text{odd}} + dK_1$. Therefore, if $t$ is odd, then $\alpha_{t-1+2j} = \alpha_{\text{even}} = u'(e_{\text{even}})/u'(e_{\text{odd}})$ and $\alpha_{t+2j} = \alpha_{\text{odd}} = u'(e_{\text{odd}})/u'(e_{\text{even}})$. But if $t$ is even, then $\alpha_{t-1+2j} = \alpha_{\text{odd}}$ and $\alpha_{t+2j} = \alpha_{\text{even}}$. In words: The MRS oscillates between two values. This is helpful for pricing the assets because each asset pays either in periods with $MRS = \alpha_{\text{even}}$ or with $MRS = \alpha_{\text{odd}}$.

Now consider an even period $t$. Asset 1 pays in odd periods $(t+1, t+3, \ldots)$. Hence, from (1), its price is given by

$$p_{1}^{\text{even}} = \alpha^{\text{odd}} d \left[ \beta + \beta^3 + \ldots \right] = \beta \sum_{j=0}^{\infty} \beta^2 j \alpha^{\text{odd}} d = \frac{\beta \alpha^{\text{odd}} d}{1 - \beta^2}$$

Here I used the fact that

$$\sum_{j=0}^{\infty} \beta^{2j} = \sum_{j=0}^{\infty} (\beta^2)^j = \frac{1}{1 - \beta^2}$$

Similarly, tree 2 pays in even periods $(t+2, t+4, \ldots)$ and its price in an even period $t$ must be

$$p_{2}^{\text{even}} = \alpha^{\text{even}} d \beta^2 \left[ 1 + \beta^2 + \ldots \right] = \beta^2 \sum_{j=0}^{\infty} \beta^2 j \alpha^{\text{even}} d = \frac{\beta^2 \alpha^{\text{even}} d}{1 - \beta^2}$$

Obviously, $\alpha^{\text{odd}} = 1/\alpha^{\text{even}} = \alpha$. Hence, the price ratio in even periods is given by

$$PR^{\text{even}} = \frac{p_{1}^{\text{even}}}{p_{2}^{\text{even}}} = \frac{\beta \alpha^{\text{odd}} d}{\alpha \alpha^{\text{even}} d} = \frac{\alpha^2}{\beta}$$

If $t$ is odd, the calculation is similar and yields

$$P_{1}^{\text{odd}} = \frac{\beta^2 \alpha^{\text{odd}} d}{1 - \beta^2}$$
$$P_{2}^{\text{odd}} = \frac{\beta \alpha^{\text{even}} d}{1 - \beta^2}$$
$$PR^{\text{odd}} = \frac{P_{1}^{\text{odd}}}{P_{2}^{\text{odd}}} = \beta \alpha^2$$

Obviously, $PR^{\text{odd}} = \beta^2 PR^{\text{even}}$. Assuming that $c^{\text{odd}} < c^{\text{even}}$, it follows that $\alpha > 1$ and $PR^{\text{even}} > 1$. The intuition is simple. Tree 1 yields fruit when consumption is low and marginal utility is high. Hence, it tends to be more valuable than tree 2. In even periods, tree 1 has the additional advantage of paying dividends one period earlier than tree 2 (in $t+1$ instead of $t+2$). Hence, $PR^{\text{even}} > 1$. However, in odd periods, tree 1 pays dividends one period later than tree 2 and $PR^{\text{odd}}$ may be less than 1.

2 Two sector model

Consider the following growth model with two capital goods.
Households: There is a single, representative household who lives forever. Preferences over consumption streams are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Households own two capital goods, $K_1$ and $K_2$. The income obtained from renting these capital goods to firms is their only source of income.

Firms: Production takes place in two sectors ($i = 1, 2$). The resource constraints for sector 1 is

$$A_1 F_1(K_{11t}, K_{12t}) + (1 - \delta) K_{1t} = K_{1t+1} + c_t$$

where $K_{ist}$ is the amount of capital of type $s$ used in sector $i$ and $K_{st} = K_{ist} + K_{2st}$ is the total amount of capital good $s$ used in both sectors. This piece of notation is important: sector $i$ uses $K_{i1}$ and $K_{i2}$. Capital good $s$ is used as $K_{is}$ and $K_{2s}$.

The resource constraint for sector 2 is similar, except that good 2 is not consumed:

$$A_2 F_2(K_{21t}, K_{22t}) + (1 - \delta) K_{2t} = K_{2t+1}$$

In each sector, a representative firm maximizes period profits. Assume that both production functions exhibit constant returns to scale.

Questions:

(a) Define a solution to the firm’s problem in each sector. Be careful to define the purchase and rental prices of the various goods consistently. Good 1 is the numeraire.

(b) State the household problem and define a solution. Remember that holding $K_2$ yields capital gains or losses in addition to rental income if $p_2$ changes over time.

(c) Define a competitive equilibrium. Make sure that the number of objects equals the number of equations.

(d) Consider the balanced growth path (where all variables grow at time-invariant rates). Which variables are constant and which variables grow? Assume log utility: $u(c) = \ln(c)$.

(e) Derive 7 equations in the following objects, which are constant on the balanced growth path: the relative price of good 2, the growth rate, the real rate of return, the rental prices for capital $(r_1, r_2)$, and $k_i$ for $i = 1, 2$, where $k_i = K_{i2}/K_{i1}$ is the input ratio in sector $s$.

(f) Consider the balanced growth effects of an increase in $A_1$. What do you expect to happen to the balanced growth rate and to prices? Explain your answer. Note: It is possible to derive the answer, but it takes some work. It is enough to explain the intuition.

2.1 Answer sketch: Two sector model

To begin, we define prices. $r_{st}$ is the rental price of capital good $s$ in terms of good 1. $p_{st}$ is the purchase price of good $s$ ($p_{1t} = 1$).

(a) The firm in sector $i$ solves

$$\max A_i F_i(K_{i1t}, K_{i2t}) p_{it} - \sum_s r_{st} K_{ist}$$

The first-order conditions are $r_s = A_i F_i(K_{i1}, K_{i2}) p_i$ for $s = 1, 2$. The right-hand-side is the value marginal product of capital good $s$ in sector $i$. A solution is a pair $(K_{i1t}, K_{i2t})$ which satisfies the 2 first order conditions.
(b) We anticipate that both capital goods must pay the same rate of return in equilibrium; call it $R$. Denote household wealth by $a_t = p_{1t}K_{1t} + p_{2t}K_{2t}$. Then the budget constraint is $a_{t+1} = R_t a_t - c_t$. The household problem is entirely standard with Euler equation $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. A solution is a sequence $(c_t, a_t)$ which satisfies Euler equation and budget constraint (and a transversality condition).

(c) A competitive equilibrium is a set of sequences $(c_t, a_t, K_{1t}, K_{2t}, r_{st}, p_{st}, R_t)$ (13 objects) which satisfy:

- 2 household conditions (see b).
- 4 firm conditions (see a).
- Definition of the rate of return:
  $$R_{t+1} = \left[(1 - \delta)p_{st+1} + r_{st+1}\right]/p_{st}; \quad s = 1, 2$$
  Giving up one unit of good $s$ today costs $p_{st}$ units of consumption. Investing the good as capital pays $\left[(1 - \delta)p_{st+1} + r_{st+1}\right]$ units of consumption tomorrow. This rate of return must be the same for both goods (2 equations).
- Goods market clearing in both sectors (given in the question).
- Capital market clearing (also given): $K_{st} = \sum_i K_{ist}$.
- Definition of $a_t$.
- The normalization $p_{1t} = 1$.

There are $2 + 4 + 2 + 2 + 2 + 1 = 14$ equations. One is redundant by Walras’ law.

(d) We know that $R$ must be constant, otherwise the consumption growth rate would not be. By the definition of $R$, this requires constant prices and rental prices. The quantities grow, all at rate $\gamma$.

(e) The Euler equation implies
$$1 + \gamma = \beta R.$$  
(2)

The definition of $R$ yields 2 additional equations
$$R = 1 - \delta + r_s/p_s.$$  
(3)

The firms’ first-order conditions are
$$r_s = A_i p_i F_{is} (K_{i1}, K_{i2})$$
Note that the marginal products (b/c of constant returns to scale) only depend on the inputs ratios $k_i = K_{i2}/K_{i1}$:
$$r_s = A_i p_i F_{is} (k_i)$$
(slightly abusing notation) (4 equations). With better notation: define \( f_i(k_i) = F_i(1, K_{i2}/K_{i1}) \). Then the firms’ FOCs become

\[
\begin{align*}
r_1 &= A_i p_i \left[ f_i(k_i) - f_i'(k_i) k_i \right] \\
r_2 &= A_i p_i f_i'(k_i)
\end{align*}
\]  

(4)

for \( i = 1, 2 \). This is entirely analogous to a model with capital and labor. Note that a higher \( k_i \) reduces \( f_i(k_i) \) but increases \( f_i(k_i) - f_i'(k_i) k_i \).

(f) Take the ratio of (4), (5) for both sectors and write this as \( r_1/r_2 = g_i(k_i) \). Note that \( g_i \) does not depend on \( A_i \) and that \( g_i'(k_i) > 0 \). From \( g_1(k_1) = g_2(k_2) \) it follows that \( k_1 \) and \( k_2 \) are positively related. Define this relationship as \( k_2 = h(k_1) = g_2^{-1}(g_1(k_1)) \). The positive relationship is not surprising. When \( r_1/r_2 \) increases, firms in both sectors substitute towards the cheaper capital good. Now consider the condition

\[
r_1 = A_1 \left[ f_1(k_1) - f_1'(k_1) k_1 \right] = A_2 f_2(k_2) = r_2/p_2
\]  

(6)

This can be written as

\[
\frac{f_1(k_1) - f_1'(k_1) k_1}{f_2(h(k_1))} = \frac{A_2}{A_1}
\]  

(7)

The LHS of (7) is increasing in \( k_1 \). It follows that a higher \( A_1 \) reduces \( k_1 \) and \( k_2 \). The intuition is that \( K_1 \) is cheaper to produce and used relative more intensively. A lower \( k_2 \) implies a higher \( r_1 = r_2/p_2 \) by (6). Therefore \( R \) and the balanced growth rate \( \gamma \) must both increase.

3 Consumption Taxes in a Growth Model

Consider the following version of the growth model. There is a single representative agent with preferences given by:

\[
\sum_{t=0}^{\infty} \beta^t \log c_t
\]

where \( c_t \) is consumption in period \( t \), and \( 0 < \beta < 1 \). The worker is endowed with one unit of time in each period but does not value leisure.

There are two production sectors. One sector produces the consumption good using a Cobb-Douglas technology:

\[
c_t = k_{ct}^\theta n_{ct}^{1-\theta}
\]

where \( k_{ct} \) and \( n_{ct} \) are capital and labor inputs to this sector at time \( t \) respectively. The other sector produces capital goods also using a Cobb-Douglas technology:

\[
i_t = A k_{it}^\eta n_{it}^{1-\eta}
\]

where \( k_{it} \) and \( n_{it} \) are capital and labor inputs to the investment sector. Feasibility requires:

\[
(1 - \delta) k_t + i_t = k_{t+1} \\
k_{ct} + k_{it} = k_t \\
n_{ct} + n_{it} = 1
\]
where $\delta$ is the depreciation rate for physical capital. Thus, we are assuming that capital is completely mobile across sectors. The initial capital stock $k_0$ is given.

(a) Define a competitive equilibrium for this economy in sequence form. Normalize the price of capital to 1.

(b) Define a steady state competitive equilibrium for this economy. Derive an equation to characterize the steady state value of the capital stock.

(c) Assume that the government places a proportional tax on consumption expenditures equal to $\tau_c$ and then simply throws away the tax revenues. How will this affect the steady state values for the capital stock, investment and consumption? Justify your answer. There is no need to resolve the model.

3.1 Answer Sketch: Consumption Taxes in a Growth Model

(a) The numeraire is capital. The price of consumption is $p_t$. The household maximizes discounted utility subject to

$$k_{t+1} = R_{t+1} k_t + w_t - p_t c_t$$

The Euler equation is

$$u'(c_t) = R_{t+1} u'(c_{t+1}) p_t / p_{t+1}$$

Firms in sector $j$ solve

$$\max p_j F_j(k_j, n_j) - r k_j - w n_j$$

First order conditions are

$$w / p_j = f_j(x_j) - f'_j(x_j) x_j$$
$$r / p_j = f'_j(x_j)$$
$$x_j = k_j / n_j$$

Competitive Equilibrium: Sequences \{c_t, k_t, k_{jt}, n_{jt}, R_t, r_t, w_t, p_t\} which satisfy:

- 2 household conditions
- 4 firm conditions
- Market clearing: Labor. $c = k_c^\theta n_c^{1-\theta}$. $k_{t+1} = A k_i^\eta n_i^{1-\eta} + (1 - \delta) k$.
- Identities: $k = k_i + k_c$. $R = 1 + r - \delta$.

(b) Steady state: A steady state consists of the same 10 variables (without the time subscripts), which satisfy the same 11 conditions. The Euler equation becomes $\beta R = 1$. The investment firm’s FOC determines the capital-labor ratio in that sector:

$$r = R - 1 + \delta = A \eta x_i^{\eta-1}$$

The market clearing condition for good $i$ implies:

$$\delta k = A k_i^\eta n_i^{1-\eta}$$
The requirement that $w/r$ is the same in both sectors yields
\[ x_c \frac{1 - \theta}{\theta} = x_i \frac{1 - \eta}{\eta} \]
Together with
\[ k = k_i + k_c = n_i x_i + (1 - n_i) x_c \]
we have an equation solving for $n_i$:
\[ k = A n_i x_i^\eta / \delta = n_i x_i + (1 - n_i) x_c \]
The solution is
\[ n_i = x_i^{1 - \eta} \delta / A \]
Hence, $k = x_i$.

(c) Consumption tax: The only change is in the household budget constraint, where prices are replaced with $(1 + \tau)p_t$. This does not affect the Euler equation or any of the other equations used in the derivation of the steady state value of $k$. The only change is that consumption falls by the amount of the tax.

Note: A number of students misinterpreted my comment to set the price of capital to 1. Not recognizing that there is a relative prices of capital versus consumption (as always in multi-sector models), these students set the rental price of capital to 1. One could actually do that, but then one would need prices for both goods (not equal to 1).

4 Asset Pricing

Consider a version of the Lucas tree model in which preferences are given by $E \sum_{t=0}^{\infty} \beta^t \ln c_t$. The state of the economy is determined by the realization of a random variable, $z$. In good times ($z = 1$), a tree gives $\bar{d}$ units of the consumption good. In bad times ($z = 0$), the tree gives $\theta \bar{d}$, where $1 > \theta > 0$. Let $\Pr(z' = 1 | z = 1) = \pi_1$ and $\Pr(z' = 0 | z = 0) = \pi_0$. In addition, suppose that there is a market for one period contingent claims. In particular, let $q(z', z)$ be the price of a unit of output to be delivered next period in state $z'$ given that the current state is $z$.

1. Define a recursive competitive equilibrium for this economy. Include a market for both trees and contingent claims.

2. Show that the equilibrium price of a tree is $\frac{\beta}{1 - \beta} \bar{d}$ in good times and $\frac{\theta \beta}{1 - \beta} \bar{d}$ in bad times (Assume that $\lim_{s \to -\infty} \beta^s E_t \{ p_{t+s} \} = 0$).

3. Derive an expression for the equilibrium contingent claim prices, $q(z', z)$, for each $\{z, z'\}$ combination.
4.1 Answer Sketch: Asset Pricing

1. Let the units of state contingent claims be denoted by \( y \). The household solves the following dynamic program

\[
v(s, y(z), z) = \max_{s', y(z') = 1, y(z') = 0} \{ \ln c + \beta E \{ v(s', y(z'), z') \} \}
\]

subject to

\[
c + ps' + q(z' = 1|z) y(z' = 1) + q(z' = 0|z) y(z' = 0) = s(p + d(z)) + y(z),
\]

where

\[
d(z) = \begin{cases} \dd, & \text{if } z = 1 \\ \theta d, & \text{if } z = 0 \end{cases}
\]

and where

\[
z' \sim \begin{bmatrix} \Pr(z' = 1|z = 1) = \pi_1 & \Pr(z' = 0|z = 1) = 1 - \pi_1 \\ \Pr(z' = 1|z = 0) = 1 - \pi_0 & \Pr(z' = 0|z = 0) = \pi_0 \end{bmatrix}
\]

A recursive competitive equilibrium is:
(a) A set of individual decision rules \( s'(s, y(z), z), y(z' = 1|s, y(z), z), y(z' = 0|s, y(z), z) \)
(b) A set of pricing functions \( p(z), q(z' = 1|z), \) and \( q(z' = 0|z) \)

such that
(a) Given \( p(z), q(z' = 1|z), \) and \( q(z' = 0|z) \) the set of individual decision rules \( s'(s, y(z), z), y(z' = 1|s, y(z), z), \) and \( y(z' = 0|s, y(z), z) \) solve the household’s dynamic program.
(b) Markets clear:

\[
\begin{align*}
s'(1, 0, z) &= 1, \\
y(z' = 1|1, 0, z) &= 0, \\
y(z' = 0|1, 0, z) &= 0, \\
c &= d(z)
\end{align*}
\]

2. FOC

\[
s' : \frac{p(z)}{c} = \beta E \{ v_1(s', y(z'), z') \} \tag{8}
\]

EC

\[
s : v_1(s, y(z), z) = \frac{1}{c} (p(z) + d(z)) \tag{9}
\]

Substitute (9) in (8) to obtain:

\[
\frac{p(z)}{c} = \beta E \left( \frac{1}{c'} (p(z') + d(z')) \right)
\]

At this point, it is useful to use time subscripts:

\[
\frac{p_t}{c_t} = \beta E_t \left( \frac{p_{t+1}}{c_{t+1}} + \frac{d_{t+1}}{c_{t+1}} \right)
\]
In equilibrium $c_t = d_t$ for all $t$. Hence,

$$\frac{p_t}{d_t} = \beta E_t \left( \frac{p_{t+1}}{d_{t+1}} + 1 \right)$$

Iterating forward and using the law of iterated expectations, it can be shown that

$$\frac{p_t}{d_t} = \frac{\beta}{1 - \beta}$$

Hence,

$$p_t = \begin{cases} \frac{\beta}{1 - \beta} d & \text{in good times} \\ \frac{\beta}{1 - \beta} \theta d & \text{in bad times} \end{cases}$$

3. FOC

$$y(z') = \begin{cases} q(z' = 1 | z) = \beta E \left\{ v_2 (s', y(z'), z') \right\} | z' = 1, \\ q(z' = 0 | z) = \beta E \left\{ v_2 (s', y(z'), z') \right\} | z' = 0 \end{cases}$$

EC

$$y(z) : v_2 (s, y(z), z) = \frac{1}{c}$$

Substitute (11) in (10) to obtain

$$q(z' = 1 | z) = \beta E \left\{ \frac{c}{c} \right\} | z' = 1, \\ q(z' = 0 | z) = \beta E \left\{ \frac{c}{c} \right\} | z' = 0$$

Note that $c = d$ in equilibrium. Using corresponding probabilities yields

$$q(z' = 1 | z = 1) = \beta \pi_1 , \quad q(z' = 1 | z = 0) = \beta \theta \left( 1 - \pi_0 \right) , \quad q(z' = 0 | z = 1) = \beta \frac{1 - \pi_1}{\theta} , \quad q(z' = 0 | z = 0) = \beta \pi_0$$

5 Taxes and government spending

Consider the following growth economy, extended to include government spending and a tax on output at rate $\tau_t$. A representative household chooses $\{c_t, k_{t+1}\}_{t=0}^\infty$ to solve the problem

$$\max \sum_{t=0}^\infty \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = (1 - \tau_t)f(k_t)$$
$k_0$ given.

Capital depreciates completely each period. The government finances exogenous spending, $g_t$, each period by levying taxes at rate $\tau_t$. Government spending does not affect private utility or production possibilities. The government budget constraint is

$$\tau_t f(k_t) = g_t$$

You may find it convenient to define government spending as a share of output by introducing the variable

$$s_t^g = g_t / f(k_t)$$

Further, note the two budget constraints imply the aggregate resource constraint

$$c_t + g_t + k_{t+1} = f(k_t)$$

or

$$c_t + k_{t+1} = (1 - s_t^g) f(k_t)$$

The household takes the time paths of the policy variables as given. Assume the functional forms: $u(c) = \ln(c)$ and $f(k) = k^\theta$, $0 < \theta < 1$.

(a) Show that there is a maximum sustainable capital stock, $k_{\text{max}}$, for this economy.

(b) Assuming that $k_0 \in (0, k_{\text{max}})$ find the steady state level of the capital stock. Assume that $\tau$ is constant over time. Note that there are no firms; households produce and consume.\footnote{You can convince yourself, however, that it would not make a difference if firms were added to the model.}

(c) Define the state of the economy and write down Bellman’s equation for the optimum problem of the household.

(d) Solve for the equilibrium consumption and investment decision rules as functions of the current state. Hint: What are reasonable guesses given log utility?

(e) Why doesn’t expected future policy affect current consumption and investment decisions?

Consider the same economic environment as above, but now allow the government to sell real one-period bonds to finance any discrepancies between spending and revenues. Let $b_{t+1}$ denote bonds sold in period $t$ at price $q_t$, which pay one unit of consumption goods in period $t+1$. Agents enter each period $t$ with capital, $k_t$, and bonds, $b_t$.

The representative household now chooses $\{c_t, k_{t+1}, b_{t+1}\}_{t=0}^\infty$ to maximize utility subject to

$$c_t + k_{t+1} + q_t b_{t+1} = (1 - \tau_t) f(k_t) + b_t$$

$k_0$ given. Assume the same functional forms before. The household takes as given the time paths of policy variables. The government chooses paths for

$$\{\tau_t, b_{t+1}\}_{t=0}^\infty$$

to finance $s_t^g$ according to:

$$q_t b_{t+1} + \tau_t f(k_t) = g_t + b_t$$

$b_0$ given.

(f) Define the state of the economy and write down Bellman’s equation for the household.

(g) Assuming that in a steady state the tax rate and government spending share are given and constant, derive expressions for steady state consumption, capital, and government debt.
5.1 Answer: Taxes and government spending

(a) It suffices to show that \( k \) is bounded, even if \( c \) and \( g \) both equal zero forever. Since

\[
k_{t+1} = k_t^\theta
\]

the corresponding steady state has \( k = 1 \). If \( k_t > 1 \) then \( k_{t+1} < k_t \).

(b) The objective function for the household problem is

\[
\sum_{t=0}^{\infty} \beta^t u([1 - \tau]k_t^\theta - k_{t+1})
\]

The first-order condition for \( k \) is:

\[
\frac{\beta^t (1 - \tau) \theta k_t^{\theta - 1}}{c_t} - \frac{\beta^{t-1}}{c_{t-1}} = 0
\]

In steady state this reduces to

\[
\beta (1 - \tau) \theta = k^{1-\theta}
\]

(c) The state is \((k_t, s_t^\theta)\). Bellman’s equation is

\[
V(k, s^\theta) = \max \ln \left((1 - s^\theta)k^\theta - k'\right) + \beta V(k', s'^\theta)
\]

(d) The first-order condition for the control \((k')\) is

\[
1/c = \beta V_k(k')
\]

The envelope condition is

\[
V_k = \left((1 - s^\theta)\theta k^{\theta - 1}\right)/c
\]

The Euler equation is therefore

\[
1/c = (1 - s^\theta')\theta k'^{\theta - 1} \beta / c'
\]

Let

\[
y(k, s^\theta) = (1 - s^\theta)k^\theta
\]

Guess \(c(k, s^\theta) = \alpha y\) and

\[
k'(k, s^\theta) = (1 - \alpha)y
\]

Use the Euler equation to solve for \(\alpha\):

\[
1/(\alpha y) = \beta \theta (y'/k')/(\alpha y')
\]

\[
\Rightarrow
1 - \alpha = \beta \theta
\]

Strictly speaking we should now also guess a value function to verify that the policy function together with \( V \) satisfies the Bellman equation. Guess:

\[
V(k, s^\theta) = E + F \ln \left((1 - s^\theta)k^\theta\right)
\]

Therefore,

\[
V'(k) = F \theta / k
\]
Apply the policy function to the right hand side of the Bellman equation:

\[ V(k, s^g) = \ln \left( (1 - s^g)ak^\theta \right) + \beta E + \beta F \left[ C_1 + \theta^2 \ln(k) \right] \]

Thus,

\[ V'(k) = F \theta / k = (1 + \beta \theta F) \theta / k \]

Therefore, \( F = 1/(1 - \beta \theta) \), which is unsurprisingly what we found above for the case without taxes – the tax just serves as a shifter of the production function. The verification step is the same as above.

(e) Future policy does not affect consumption decisions because of log utility.

(f) The state is now \((b_t, k_t, s_t^g)\). The Bellman equation is:

\[ V(k, b, s^g) = \max \ln \left( (1 - \tau)f(k) + b - qb' - k' \right) + \beta V(k', b', s^g') \]

The FOCs are \( \beta V_k(.) = 1/c \), \( \beta V_b(.) = q/c \). The envelope conditions are:

\[
\begin{align*}
V_k &= (1 - \tau)f'(k)/c \\
V_b &= 1/c
\end{align*}
\]

Combining them yields:

\[ \beta(1 - \tau)f'(k')/c' = 1/c \]

and

\[ q/c = \beta/c' \]

The first implication is that both assets must yield the same rate of return:

\[ 1/q = (1 - \tau)f'(k') \]

Furthermore, the Euler equation implies that the steady state interest rate equals the discount rate:

\[ \beta(1 - \tau)f'(k) = 1 \]

This is a very important finding that arises all the time in infinite horizon models and greatly simplifies the analysis. Therefore \( \beta = q \) and

\[ k = \{\beta \theta(1 - \tau)\}^{1/(1 - \theta)} \]

The level of debt follows from the government budget constraint:

\[ b = (\tau - s^g)k^\theta / (1 - \beta) \]

Finally, the level of consumption can be obtained from the household’s budget constraint,

\[ c = (1 - \tau)f(k) - k + (1 - \beta)b \]