Do problems 6 and 7 on problem set 8.

1. There are two firms \((A, B)\). Each firm uses inputs of \(x_1\) and \(x_2\) to produce output of good \(q\). Let \(\{x_i^i, q^i\}\) denote firm \(i\)’s input and output vector and assume they have the following technology:

\[
q^A_i \leq 10\left(\left(x_1^A\right)^{1/4} + \left(x_1^A\right)^{1/4}\right)^\alpha; \quad q^B_i \leq 5\left(\left(x_1^A\right)^{1/4} + \left(x_1^A\right)^{1/4}\right)^\alpha; \quad \alpha > 0
\]

(a) Assuming \(\alpha < 4\), find each firm’s profit maximizing decisions and its maximum profit function, \(\pi^i(p, w_1, w_2)\). Also, find the “aggregate” supply and factor demand curves by adding together the supply and demand curves for the two firms.

(b) **Next**, consider the “industry” production function, \(q^T\left(x_1^T, x_2^T\right)\) obtained by solving the following problem:

Given \(x_1^T, x_2^T\), \(\max\left(q^A_i + q^B_i\right)\) subject to the resource constraints: \(x_1^A + x_i^B \leq x_i^T, \ i = 1, 2\).

i. For \(\alpha = 4\), what is the industry production function. Will both firms be used to produce output?

ii. For \(\alpha < 4\), find the industry production function. Will both firms be used to produce output?

(c) **Assuming \(\alpha < 4\), find the** aggregate supply and input demands \(\left(q^T, x_1^T, x_2^T\right)\) that maximize profits for the aggregate production function derived in part (b), and find the maximum (aggregate) profits.

(d) **Show** that the aggregate maximum profit function, and the corresponding output supply and input demands, are just the sum of the individual firm’s rules (e.g., \(\pi^T = \pi^A + \pi^B\)).

(e) An aggregate netput vector \(y^T\) is said to be efficient if there does not exist a technologically feasible vector \(y’\) (in the aggregate production set) such that \(y’ \geq y^T\), \(y’ \neq y^T\) \{in this case, the netput vector is \(y = \{q, -x_1, -x_2\}\)\}. Use the previous parts to argue that individual profit maximization leads to an aggregate production vector that is efficient.

2. (Generalization of Problem 2): There are \(J\) firms; each firm produces the same good \((q)\) using the same set of inputs \(\left(z_1, \ldots, z_L\right)\). The production set for each firm is strictly concave and given by:

\[
q_j \leq F^j\left(z_1, \ldots, z_L\right)
\]
(a) A central planner, with given amounts of each input, wishes to maximize output of good $q$: the optimization problem is written as:

$$L = \sum_j q_j + \sum_i \gamma_i \left(z_i^T - \sum_j z_j^i\right) = \sum_j F^j \left(z_j^i, \ldots, z_L^i \right) + \sum_i \gamma_i \left(z_i^T - \sum_j z_j^i\right)$$

where $z_i^T$ is the availability of each input. The Lagrangean, evaluated at the solution, gives the industry production function: $Q^* \left(z_1^T, \ldots, z_L^T\right) = L \left(z_1^T, \ldots, z_L^T ; z_j^* \gamma_j^* \right) \Rightarrow \left(z_j^* \gamma_j^* \right)$ is the solution vector.

i. Must a solution exist for this problem? If one does, write the FOC for the optimization problem and interpret. Are the SOSC satisfied?

ii. What is the meaning of the Lagrangean multipliers?

iii. What do the derivatives of $Q^* \left(z_1^T, \ldots, z_L^T\right)$ with respect to $z_j^T$ represent?

iv. Consider the simple case of 3 firms, and one input, with technologies given by:

**Firm 1:** $q_1 \leq 10z_1^{1/2}$;  **Firm 2:** $q_2 \leq 20z_2^{1/2}$;  **Firm 3:** $q_3 \leq 5 \left( \left(z_3 + 1\right)^{1/2} - 1 \right)$

(since there is only one input, $z_j$ denotes firm $j$’s use of that input).

Derive the industry production function; pay attention to corner solutions.

(b) Consider the case of competitive profit maximizing firms, with the technology of part (a), i.e., $q_j \leq F_j \left(z_j^i, \ldots, z_L^i \right)$. All firms face price vectors $(p, w_1, \ldots, w_L)$, where $w_j$ is the input price.

i. Must profit maximizing solutions exist? Assuming they do, write the FOC for the profit-maximizing problem for each firm. Are the SOSC satisfied?

ii. Assuming a solution to the profit maximization problem, show that the solution to this problem and the solution to the central planner’s problem (part a) are identical if $(w_j / p) = \gamma_j^*$. Does this depend upon the solutions being interior?

iii. Derive the firms’ supply curves and the industry supply curve for the technology of part (aiv) above. What is the relationship between the industry supply curve – derived from profit maximization – and the efficient industry production function $Q^*$ - derived by the central planner? Answer for both the general case and the specific case.

3. Let $c(w, q)$ denote the cost function of a competitive firm, where $q$ is output and $w$ is the vector of input prices. Assume that it takes the following form:

$$c(w, q) = \begin{cases} 4w_1 + q^{3/2} \sqrt{w_1w_2} & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases}$$
(a) Find the firm's profit-maximizing supply function. Now suppose that \( w_1 = w_2 = 1 \), and let \( p \) represent output price. What is the firm's optimal output if \( p = 5 \)? What about when \( p = 2 \)?

Assume that there is free entry in this competitive market, and that input prices are \( w_1 = w_2 = 1 \). What is the long-run supply correspondence for this industry?

(b) The output of this industry is demanded by 800 consumers, each with indirect utility function \( V_i = \omega_i - 10p + p^2/2 \), where \( i \) indexes consumers and \( \omega \) denotes income (measured in units of a numeraire good). Input prices are still assumed fixed at \( w_1 = w_2 = 1 \). Determine the long-run equilibrium (including the long-run number of firms) in this market.

(c) What would happen to your answer to part (c) if there were 850 consumers? Is there any problem with existence of the equilibrium? Explain.

4. Consider a competitive industry in long run equilibrium. All firms are identical, each with cost function \( C(w,q) \), where \( q \) denotes the output of one firm, and \( w \) is the vector of input prices. This cost function displays a U-shaped average cost and a strictly increasing marginal cost. The (downward sloping) market demand for this industry is written as \( x(p,\alpha) \), where \( p \) denotes the price for the industry output and \( \alpha \) is a shift parameter.

All input prices except \( w_1 \) are exogenous. However, this industry is the only user of input 1, and the market supply of this input is given by \( S(w_1) \), where \( S'(w_1) > 0 \).

The industry long-run equilibrium is characterized by the values of \( \{ p^*, q^*, J^*, w_1^* \} \), where \( J \) denotes the number of firms. [Strictly speaking \( J \) is an integer, but you can ignore that and treat \( J \) as a real number].

(a) Write down the system of equations that define the long-run equilibrium. Briefly discuss the rationale behind each of the equations. Also, show how an increase in demand (an increase in \( \alpha \), since \( \partial x / \partial \alpha > 0 \)) affects the equilibrium.

(b) Now assume \( w_1 \) is exogenous (i.e., \( S(w_1) \) is infinitely elastic). Use comparative statics on the system of equations derived in (a) – with \( w_1 \) constant - to determine the impact on the long run equilibrium of an increase in an input price \( w_k \), where input \( k \) is an inferior input [Recall that an input is said to be inferior if the cost-minimizing input demand is negatively related to output, i.e., \( \partial x^*_k(w,q)/\partial q \leq 0 \)]. Specifically, determine the signs of \( \partial p^*/\partial w_k \), \( \partial q^*/\partial w_k \) and \( \partial J^*/\partial w_k \).