1. **Homothetic Preferences**

Assume the utility function \( U(\bar{x}) \) is homothetic, so that \( \frac{U_i(\lambda \bar{x})}{U_j(\lambda \bar{x})} = \frac{U_i(\bar{x})}{U_j(\bar{x})}, \forall \lambda > 0, \forall \bar{x} \geq 0 \).

Assuming an interior solution:

a) What can you conclude about the functional forms of the expenditure function, indirect utility function and the Marshallian and Hicksian demand functions?

b) Consider the cross partial derivatives of the Marshallian demand functions \( \frac{\partial x_i}{\partial p_j} \). What can you conclude about the relationship between \( \frac{\partial x_i}{\partial p_i} \) and \( \frac{\partial x_j}{\partial p_i} \), \( i \neq j \)? Prove your answer.

2. **Cardinality vs. Ordinality**

Suppose there are two people, Bill and Jane:

Bill’s utility function: \( U = f(x) \), with expenditure function \( e(p,u) \)

Jane’s utility function: \( U = G(x) = H(f(x)); H' > 0 \)

b) Compare Jane’s indirect utility function, Marshallian and Hicksian demands to those of Bill.

c) For the special case of \( f(x) = 5 \ln(x_1) + 10 \ln(x_2) \), and \( H = e' \) (where \( e \) is the natural base), find the expenditure function, indirect utility function, and Hicksian and Marshallian demands for both Bill and Jane.

3. **Stone-Geary utility function**

Assume that, for the 2-good case, the utility function has the so-called Stone-Geary form:

\[
u(x_1, x_2) = (x_1 - \gamma_1)^\alpha (x_2 - \gamma_2)^\beta\]

a) Explain why we should have \( \alpha > 0 \) and \( \beta > 0 \), and, without loss of generality, why it is possible to put \( \alpha + \beta = 1 \). (For the remainder of the problem, maintain these conditions).

b) Standard theory does not restrict the sign of parameters \( \gamma_1 \) and \( \gamma_2 \). Here, however, assume that \( \gamma_i \geq 0 \), \( i = 1, 2 \). Given that, define an appropriate consumption set for these preferences and provide a graphical illustration.

For the remainder of this problem, assume that an interior solution applies.

c) Set up and solve the UMP (utility maximizing problem), and derive the indirect utility function. Verify that the indirect utility function satisfies the appropriate homogeneity property.

d) Set up and solve the EMP (expenditure minimizing problem), and derive the expenditure function. Verify that the expenditure function satisfies the appropriate curvature property.
e) Verify that the indirect utility function can be alternatively obtained by inverting the expenditure function, and, correspondingly, that the indirect utility function can be obtained by inverting the expenditure function. Briefly explain why this procedure is legitimate.

4. Consider the choice problem of a consumer with a linear budget constraint and strictly convex preferences defined on \( \mathbb{R}_+^3 \), and the standard notation whereby \( p_i \) denote prices, \( w \) is income, and \( u \) is a utility level. Suppose that the Marshallian demand functions for the first two goods are:

\[
x_i^m(p_1, p_2, p_3, w) = \frac{3wp_1^{-3/4}}{4\left[p_1^{3/4} + p_3^{3/4}\right]^2}; \quad x_2^m(p_1, p_2, p_3, w) = \frac{w}{4p_2}
\]

a) Find the Marshallian demand for good 3.

b) Was the utility function from which these preferences were derived homothetic? Explain.

c) Are goods 2 and 3 substitutes or complements? What about goods 1 and 2? How can you tell?

5. Consider the choice problem of a consumer with a linear budget constraint and strictly convex preferences defined on \( \mathbb{R}_+^3 \), and the standard notation whereby \( p_i \) denote prices, \( w \) is income, and \( u \) is a utility level. Suppose that the Marshallian demand functions for the first two goods and the Hicksian demand function for the third good are:

\[
x_i^H(\tilde{p}, w) = \frac{u_{1/2}\left(p_1^{1/4} + p_2^{1/4}\right)p_i^{-3/4}p_3^{1/2}}{2}; \quad x_1^H(\tilde{p}, w) = \frac{u_{1/2}\left(p_1^{1/4} + p_2^{1/4}\right)p_2^{-3/4}p_3^{1/2}}{2}; \quad x_3^H(\tilde{p}, w) = \frac{w}{2p_3}
\]

a) Was the utility function from which these preferences were derived homothetic? Explain.

b) Find the indirect utility function and the Marshallian and Hicksian demands for all three goods.

6. **Duality** – Recovering preferences. Given the following expenditure functions, find the corresponding indirect utility function \( V(\tilde{p}, w) \) and the direct utility function \( u(\tilde{x}) \). Also, indicate whether the utility function you derive is quasi-concave, and whether it is homothetic. Finally, indicate for which good, if any, the income elasticity of demand exceeds one.

a) \( e(\tilde{p}, u) = \ln u \cdot \left(p_1^{3/4}p_2^{1/4}\right); \quad u \geq 0; \quad p \gg 0 \)

b) \( e(\tilde{p}, u) = u\left(p_1p_2\right)^{1/2} + 2u^{1/2}p_3 \)

c) \( e(\tilde{p}, u) = u^3\left[p_1^\kappa + p_2^\kappa\right]^{1/\kappa} \)

{What is the domain for \( \kappa \) that guarantees this is a valid expenditure function?}
d) \[ e(\hat{p}, u) = \text{Min} \left( \frac{p_1}{2}, p_2 \right) u^3; \quad u \geq 0; \quad p \gg 0 \]

e) \[ e(\hat{p}, u) = \text{Min} \left( 2(p_1 p_2)^{1/2}, p_3 \right) u^{1/2}; \quad u \geq 0; \quad p \gg 0 \]

f) Suppose a positive monotonic transformation \( H(u), \quad H' > 0 \) had been performed on each of the utility functions dual to the expenditure functions in a-e. How would that change the expenditure function?

7. **Duality and Convexity**

(a) Consider the utility function \( U(x_1, x_2) = \text{Max} \left( \left[ x_1 x_2^2 \right], \left[ x_1^3 x_2 \right] \right) \).

i. Is this function quasi-concave (are preferences convex)?

ii. Find the expenditure function and indirect utility function for these preferences.

iii. Use either the expenditure function or the indirect utility function to derive the quasi-concave utility function dual to the expenditure function.

iv. Is there more than one non-quasi-concave utility function that could give rise to this expenditure function? If so, do all these utility functions represent the same preferences?

(b) Consider the utility function: \( U = 9x_1 x_2 - (x_1)^2 - (x_2)^2 \). Restrict the consumption space so that consumption is non-negative and utility is non-negative.

i. In the specified domain, are preferences convex?

ii. In the specified domain, do preferences satisfy the non-satiation axiom? Are they monotonic?

iii. Assuming \( p \gg 0 \), derive the demand functions and the expenditure function.

iv. Can you recover the original preferences from the expenditure function? If not, what portion of preferences can you recover?

(It should help you to graph indifference curves for the function, paying careful attention to where the indifference curves are positively sloped).

8. **WARP vs. SARP**

In a 3 good world suppose the consumer makes the following choices:

At \( \tilde{p}^0 = (6,8,10) \) she buys: \( \tilde{x}^0 = (4,5,3) \);

At \( \tilde{p}^1 = (8,10,6) \) she buys: \( \tilde{x}^1 = (5,4,3) \);

At \( \tilde{p}^2 = (8,7,10) \) she buys: \( \tilde{x}^2 = (3,4,4) \);

(a) Using WARP, with this data is \( \tilde{x}^0 \) revealed preferred to \( \tilde{x}^1 \)? Explain.
(b) Using WARP, find the values of $A$ for which: (i) $\tilde{x}^1$ is revealed preferred to $\tilde{x}^2$; (ii) $\tilde{x}^2$ is revealed preferred to $\tilde{x}^1$; (iii) the data does not provide information about ranking the two bundles using WARP; and (iv) the data is inconsistent with WARP.

(c) Using WARP, find the values of $A$ for which: (i) $\tilde{x}^2$ is revealed preferred to $\tilde{x}^0$; (ii) $\tilde{x}^0$ is revealed preferred to $\tilde{x}^2$; (iii) the data does not provide information about ranking the two bundles using WARP; and (iv) the data is inconsistent with WARP.

(d) Are there any values of $A$ for which this data satisfies WARP but violates transitivity? If so, what are they? Which assumption is stronger – WARP or SARP?

10. In a 3 good world, suppose you have the following Marshallian demand curves for goods 1 and 2:

$$
\begin{align*}
    x_1 &= w \frac{4 p_1 \left( p_2^{1/3} + p_3^{1/3} \right)^3}{p_2 p_3}, \\
    x_2 &= \frac{\alpha p_1^{1+\beta} \left( p_2^{1/3} + p_3^{1/3} \right)^2}{p_2^{1+\beta} p_3^{2/3}}, \quad \alpha > 0, \beta > 0
\end{align*}
$$

Assume $w$ is such that an interior solution holds.

(a) Find the Marshallian demand for good 3.

(b) What must the values of $\alpha$ and $\beta$ be to satisfy the integrability conditions?