Independence

Definition

Two events are said to be independent if

$$P(A \cap B) = P(A)P(B) \tag{4}$$

- If $P(B) > 0$ (or $P(A) > 0$), this can be written in terms of conditional probability as

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

- The events A and B are independent if knowledge of B does not affect the probability of A.

Independence

Example

- Suppose that two coins are tossed once.
- The sample space is:

$$\Omega = \{HH, HT, TH, TT\}$$

- Let A be the event the first coin results in a head, B be the event that the second coin results in a tail, and C the event that both are tails.
- Find the probabilities of A, B, and C.

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{1}{4}$$

- Now find the probability of the intersections of A, B and C.

$$P(A \cap B) = \frac{1}{4}, \quad P(A \cap C) = \emptyset, \quad P(B \cap C) = \frac{1}{4}$$

- Now find $P(B|A)$ and $P(B|C)$:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

- This is the same as $P(B)$ so A and B are independent.

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

- Roll a red die and a green die.
- Let $A = \{4 \text{ on the red die}) $ and $B = \{\text{sum of the dice is odd}\}$.
- Find $P(A)$, $P(B)$, and $P(A \cap B)$.
- Are A and B independent?

There are 36 points in the sample space.

Green (A) Red (D)	1	2	3	4	5	6
1	1 1	1 2	1 3	1 4	1 5	1 6
2	2 1	2 2	2 3	2 4	2 5	2 6
3	3 1	3 2	3 3	3 4	3 5	3 6
4	4 1	4 2	4 3	4 4	4 5	4 6
5	5 1	5 2	5 3	5 4	5 5	5 6
6	6 1	6 2	6 3	6 4	6 5	6 6

- The probability of A is $6/36 = 1/6$.
- The probability of B is $18/36 = 1/2$
- The probability of $A \cap B$ is $3/36 = 1/12$.
- To check for independence multiply $P(A)$ and $P(B)$ as follows

$$P(A)P(B) = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12} = P\left(A \cap B\right)$$

- The events A and B are thus independent.

More General Definition of Independence

- The events $A_1, A_2, \ldots, A_n$ are said to be independent if

$$P\left(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}\right) = \prod_{j=1}^{k} P\left(A_{i_j}\right)$$

  for any subset $\{i_1, i_2, \ldots, i_k\}$ of the integers $\{1, 2, \ldots, , n\}$.
- If all the $P(A_i)$ are positive, we can rewrite this in terms of conditional probability as

$$P\left(A_j | A_{i_1}, A_{i_2}, \ldots, A_{i_k}\right) = P\left(A_j\right)$$

  for any j and $\{i_1, i_2, \ldots, i_k\}$ such that $j \notin \{i_1, i_2, \ldots, i_k\}$.

Additive Law of Probability

- The probability of the union of two events A and B is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A and B are mutually exclusive events, $P(A \cap B) = \emptyset$ and

$$P(A \cup B) = P(A) + P(B)$$

- This is Axiom 3 for probability defined for a discrete sample space or property two of a probability measure.
- For three events, A, B and C we find

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\ &\quad - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$